Regular paper

QUASI-STATIC CLOSED-FORM GREEN'S FUNCTIONS OF A HORIZONTAL HERTZIAN DIPOLE IN HOMOGENOUS SOIL

Mircev Miroslav[1], Ognenka GAVRILOSKA,
Vesna ARNAUTOVSKI-TOSEVA[2] and Leonid GRCEV[3]

Abstract: Calculation of electric field vector of a Hertzian dipole is the fundamental step in many moment method based electromagnetic models for analysis of conductors in presence of ground. In this paper we will investigate the possibilities for approximate calculation of E_x electric field component from a horizontal Hertzian dipole in homogenous soil by using quasi-static closed-form approximation of Green's functions. By this it would be possible to replace time consuming direct numerical integration of Sommerfeld-type integrals. Also, some numerical results will be presented.

Keywords: Electromagnetic fields, Quasi-static Sommerfeld intgal, Hertzian dipole, Homogenous ground.

INTRODUCTION

The electric field due to horizontal electric (Hertzian) dipole (HED) within ground is represented by Green's functions, which take into account the influence of the air-ground interface and the electric parameters of ground as finitely conducting medium. Together with the source and image free-space terms, the corresponding Green's function consists of a term represented by Sommerfeld integral. Because of its computationally inefficiency for exact numerical integration (numerical difficulties due to oscillations, divergent behavior and singularities), Sommerfeld integrals have been studied extensively during last decades. Various numerical and analytical techniques have been developed in order to obtain accurate, fast and efficient exact numerical or approximate results.

The electromagnetic full-wave model developed for analysis of grounding systems is based on rigorous formulations derived from the full set of the Maxwell's equations, on the theoretical background of antenna analysis. The detailed description of this model, developed for uniform soil is given in [1]. The mathematical model is solved by using the metod of moments. When calculating the impedance matrix, this model involves calculation of the electric field vector of a Hertzian dipole which is assumed as an elementary source. This procedure involves numerical calculation of the so called exact formulation for the fields of a Hertzian dipole by direct numerical integration of Sommerfeld-type integrals. Although these days computer effiency is very high, the calculation of Sommerfeld integrals is still very time consuming step. Looking for computationally more efficient solution, it is of great interest to investigate possibilities to apply more approximate but more computationally efficient methods. The main objective is to avoid time-consuming numerical integration of Somerfeld-type in tegrals when calculating electric fields for quasi-static and near field ranges. The calculation of such fields is starting point in the mathematical model of grounding system, for practical lightning studies for frequency spectrum up to several MHz.

ELECTRIC FIELD OF A HERTZIAN DIPOLE

Exact formulation

Consider a time-harmonic Hertzian dipole ($e^{j\omega t}$ time dependence is assumed and superssed) of current moment $Idl' = 1$, located in homogenous soil in orientation defined along x- axis, as presented on figure 1.

The air (characterised by permittivity ε_0 and permeability μ_0) occupies the upper half-space ($z > 0$) and is referred as medium '2', and the soil (characterised by relative permittivity ε_r, permeability μ_0 and conductivity σ_1) occupies the lower half-space ($z < 0$) and is referred as medium '1'. Let the position of the Hertzian dipole is defined by a vector $\boldsymbol{r}' = (x', y', z')$, and the observation point is defined by a vector $\boldsymbol{r} = (x, y, z)$.

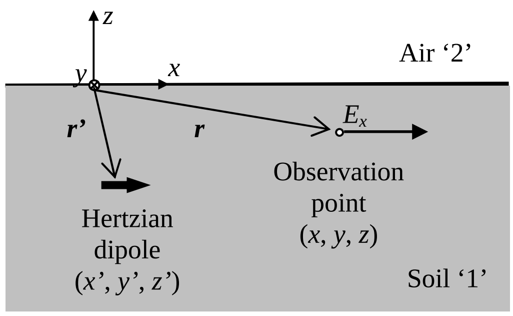

Fig.1 – *A Hertzian dipole in homogenous soil.*

We will focus our interest to the evaluation of the x- and y- components of the electric field vector:

$$E_x(\boldsymbol{r}) = -\frac{\partial}{\partial x}\phi(\boldsymbol{r}) - j\omega A_x(\boldsymbol{r}) . \qquad (1a)$$

which is derived from the x- component of the vector potential A_x and the scalar potential ϕ. The vector and scalar potentials of Hertzian dipole are given by [2]:

1 Faculty of electrical engineering, University Ss. Cyril and Methodius, Skopje, 1000 Karpos II bb, Macedonia, E-mail: miki_juve@yahoo.com
2 Faculty of electrical engineering, University Ss. Cyril and Methodius, Skopje, 1000 Karpos II bb, Macedonia, E-mail: atvesna@etf.ukim.edu.mk
3 Faculty of electrical engineering, University Ss. Cyril and Methodius, Skopje, 1000 Karpos II bb, Macedonia, E-mail: lgrcev@etf.ukim.edu.mk

$$A_x(\boldsymbol{r}) = G_A^{xx}(\boldsymbol{r}) I d1' \quad (2)$$

$$\phi(\boldsymbol{r}) = G_\phi(\boldsymbol{r}) q d1' = G_\phi(\boldsymbol{r}) \frac{-1}{j\omega}\frac{dI}{dx'} d1' , \quad (3)$$

where G_A^{xx} is the *x*-component of the magnetic vector potential dyadic Green's function of the current dipole, and G_ϕ is the electric scalar potential Green's function of a single charge associated with the horizontal Hertzian dipole. The potentials are related via the Lorentz gauge, and the current I and charge q are related via the continuity equation [2]. Above Green's functions are derived firstly in the spectral domain, afterwards they are obtained in the space domain by the following expressions:

$$G_A^{xx} = \frac{\mu_0}{4\pi}\left[G_{11} + \frac{k_1^2 - k_0^2}{k_1^2 + k_0^2} G_{12} + U'_{11}\right] \quad (4)$$

$$G_\phi^{x} = \frac{1}{4\pi\underline{\varepsilon}}\left[G_{11} + \frac{k_1^2 - k_0^2}{k_1^2 + k_0^2} + k_1^2 V'_{11}\right] \quad (5)$$

where,

$$G_{11} = \frac{e^{-jk_1 r_1}}{r_1},\ r_1 = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2} \quad (6)$$

$$G_{12} = \frac{e^{-jk_1 r_2}}{r_2}, r_2 = [(x-x')^2 + (y-y')^2 + (z+z')^2]^{1/2} \quad (7)$$

are Green's fuctions of the point source and its image in an infinite homogenous soil: direct term and image term. Terms U'_{11} and V'_{11} are Sommerfled-type integrals which generally cannot be integrated analytically:

$$U'_{11} = \int_0^\infty [\frac{2}{\gamma_1 + \gamma_0} - \frac{2k_1^2}{\gamma_1(k_1^2 + k_0^2)}] e^{-\gamma_1|z+z'|} J_0(\lambda\rho)\lambda d\lambda \quad (8)$$

$$V'_{11} = \int_0^\infty [\frac{2}{\gamma_1 k_0^2 + \gamma_0 k_1^2} - \frac{2}{\gamma_1(k_1^2 + k_0^2)}] e^{-\gamma_1|z+z'|} J_0(\lambda\rho)\lambda d\lambda \quad (9)$$

In the above expressions γ_1 and γ_0 are functions of complex λ variable:

$$\gamma_1 = (\lambda^2 - k_1^2)^{1/2} \qquad \gamma_0 = (\lambda^2 - k_0^2)^{1/2} \quad (10)$$

with $k_0^2 = \omega^2 \mu_0 \varepsilon_0$ and $k_1^2 = \omega^2 \mu_0 \varepsilon_0 \underline{\varepsilon_r}$ as the propagation constants in air and soil respectively, and $\underline{\varepsilon_r} = \varepsilon_r - j\sigma/(\omega\varepsilon_0)$ as the complex relative dielectric constant of the soil.

The above expressions (4 - 5) and (6 - 9) represent so called exact formulation of the fields due to a horizontal Hertzian dipole in homogenous soil. This formulation may be used without limits in frequency domain, and enables computation of near and far fields. However, this formulation is found to be computationally time consuming and inefficient, because the involved Sommerfeld integrals has to be solved by appropriate numerical integration.

Qusi-static approximation of U'_{11} and V'_{11}

According to the quasi-static formulation given in [3] which is valid generally in range $0 \le k_0\rho = 1$ which correspons to radial distance $\rho = w_{AIR}/(2\pi)$, the quasi-static approximation is determined by the criteria $k_0^2 \to 0, \gamma_0 \to \lambda$. By this we may define the quasi-static approximation of Sommerfeld-type integrals U'_{11} and V'_{11} in (4-5) as:

$$U'_{11QS} = 2\int_0^\infty [\frac{1}{\gamma_1 + \lambda} - \frac{1}{\gamma_1}] e^{-\gamma_1|z+z'|} J_0(\lambda\rho)\lambda d\lambda \quad (11)$$

$$V'_{11QS} = \frac{2}{k_1^2}\int_0^\infty [\frac{1}{\lambda} - \frac{1}{\gamma_1}] e^{-\gamma_1|z+z'|} J_0(\lambda\rho)\lambda d\lambda \ . \quad (12)$$

According to the analytical closed-form solution:

$$C = \int_0^\infty \frac{e^{-\gamma_1|z+z'|}}{\gamma_1} J_0(\lambda\rho)\lambda d\lambda = \frac{e^{-\gamma_1 R}}{R}, R = \sqrt{\rho^2 + |z+z'|^2} \quad (13)$$

integrals (11) and (12) may be rewritten as:

$$U'_{11QS} = 2\int_0^\infty \frac{\lambda}{\gamma_1 + \lambda} e^{-\gamma_1|z+z'|} J_0(\lambda\rho) d\lambda - 2C \quad (14)$$

$$V'_{11QS} = \frac{2}{k_1^2}\int_0^\infty [e^{-\gamma_1|z+z'|} J_0(\lambda\rho) d\lambda - \frac{2}{k_1^2} C \ . \quad (15)$$

On the basis of the closed-form solution (13), our main objective is to obtain complete closed-form solution of quasi-static approximations of integhrals U'_{11QS} and V'_{11QS}.

Our approach is theoretically based on the quasi-static complex-image theory of Banister [4], which is proved to be useful in determining the quasi-static fields of antennas located near the earth's surface.

If we generalize the Sommerfeld-type integrals in (14) and (15) as:

$$S_0 = \int_0^\infty f(\lambda) e^{-\gamma_1|z+z'|} J_0(\lambda\rho)\lambda d\lambda \quad (16)$$

our objective is to approximate the complex function *f*(λ) by an exponential function as:

$$f(\lambda) \approx e^{-\lambda d} \ . \quad (17)$$

where $d=2/(jk_1)$ is so called complex depth.

Utilizing the image complx theory formulation [5] we may approximate the fraction term in (14) as:

$$\frac{2\lambda}{\gamma_1 + \lambda} \approx 1 - e^{-\lambda d} \ . \quad (18)$$

Now we may rewrite (14) as:

$$U'_{11QSapprox} \approx A - B - 2C \quad (19)$$

where,

$$A = \int_0^\infty e^{-\gamma_1|z+z'|} J_0(\lambda\rho) d\lambda \quad (20)$$

$$B = \int_0^\infty e^{-\lambda d} e^{-\gamma_1 |z+z'|} J_0(\lambda\rho) d\lambda \,. \tag{21}$$

In order to obtain closed-form solution of integrals *A* and *B* we introduce the following approximation of complex function $\gamma_1(\lambda)$ that is linearly related to λ through

$$\gamma_1 = \sqrt{\lambda^2 - k_1^2} \approx c_1\lambda + c_2 \,. \tag{22}$$

Here, c_1 and c_2 are complex constants obtained by formulas depending on electromagnetic properties of the medium and frequency:

$$c_1 = \frac{\sqrt{T_0^2 - k_1^2}}{T_0} - \frac{jk_1}{T_0} \text{ and } c_2 = jk_1 \tag{23}$$

where T_0 is determined from the condition >√ε·k_0.

By this approximation, it is obtained:

$$A_{approx} \approx \int_0^\infty e^{-|z+z'|(c_1\lambda + c_2)} J_0(\lambda\rho) d\lambda = \frac{e^{-c_2|z+z'|}}{\sqrt{\rho^2 + (c_1|z+z'|)^2}} \tag{24}$$

$$B_{approx} \approx \int_0^\infty e^{-c_2|z+z'|} e^{\lambda(c_1|z+z'|+d)} J_0(\lambda\rho) d\lambda = \frac{e^{-c_2|z+z'|}}{\sqrt{\rho^2 + (c_1|z+z'|+d)^2}} \,. \tag{25}$$

Similarly, the quasi-static integral V'_{11QS} given by (15) may be obtained in closed-form by:

$$V'_{11QSapprox} \approx \frac{2}{k_1^2}(B - C) \,. \tag{26}$$

Finally we may define approximate closed-form Green's functions as:

$$G^{xx}_{AQSapprox} = \frac{\mu_0}{4\pi}\left\{G_{11} + \frac{k_1^2 - k_0^2}{k_1^2 + k_0^2} G_{12} + U'_{11QSapprox}\right\} \tag{27}$$

$$G_{\phi QSapprox} = \frac{1}{4\pi\underline{\varepsilon}}\left\{G_{11} + \frac{k_1^2 - k_0^2}{k_1^2 + k_0^2} G_{12} + k_1^2 V'_{11QSapprox}\right\} . \tag{28}$$

Since above relations do not involve direct numerical computation of Sommerfeld-type integrals it is expected that they arenumerically more efficient. However, their validity and application have to be verified in order to be used in practical grounding analysis, in relation to real soil parameters and for the frequency range of interest.

The results will be also compared with the results obtained by using modified image approximation presented in [6] where:

$$G^{xx}_{AIMG} = \frac{\mu_0}{4\pi} G_{11} \tag{29}$$

$$G_{\phi IMG} = \frac{1}{4\pi\underline{\varepsilon}}\left\{G_{11} + \frac{k_1^2 - k_0^2}{k_1^2 + k_0^2} G_{12}\right\} . \tag{30}$$

NUMERICAL RESULTS

In order to verify the accuracy and to determine the criteria for utilization of approximate formulations (19) and (26) we have performed a set of numerical examples wherefrom we present some of the results.

An *x*- oriented Hertzian dipole is assumed to be positioned at location (0, 0, –1 m). The soil is assumed to be homogenous and characterized by relative dielectric constant $\varepsilon_r = 10$ and permeability μ_0 and $\sigma = 0.01$ S/m. The observation points are defined in the source *x-y* plane $z = z' = -1$ m. Distance range is assumed radially with respect to the wavelength in ground $w_{GR} = w_{AIR} / |\underline{\varepsilon}_r|$. At 1 MHz the wavelength in ground is $w_{GR} \approx 20$m.

In order to determine the accuracy of approximative formulations (19) and (26) with respect to U'_{11} we have calculated the error (in %) as follows:

$$erU = \left|\frac{U'_{11} - U'_{11QSapprox}}{U'_{11}}\right| \cdot 100 \,. \tag{31}$$

To illustrate the results we have calculated error *erU* when calculating U'_{11} at near field distances in range <0.125·w_{GR}. The results are given in table 1.

Table 1 – *Error in % obtained by using approximate formula for* U'_{11}

ρ (*m*)	ρ (w_{GR})	U'_{11}	$U'_{11QSapprox}$	*erU* (%)
0.5	0.025	-0.256228 +j 0.142427	-0.284106 +j 0.155403	10.5
1	0.05	-0.220112 +j 0.136064	-0.25365 +j 0.14177	13.1
1.5	0.075	-0.175733 +j 0.12683	-0.212267 +j 0.1287	16.8
2	0.1	-0.133119 +j 0.115951	-0.168561 +j 0.117426	20.1
2.5	0.125	-0.0966254 +j 0.10437	-0.128385 +j 0.106884	22.3

Table 2 – *Error in % obtained by using approximate formula for integral A*

ρ (*m*)	ρ (w_{GR})	*A* (20)	*A* $_{approx}$(24)	*erA* (%)
0.5	0.025	3.51643 +j 10.5849	2.76266 +j 9.64334	10.8
1	0.05	3.37866 +j 9.63991	2.79989 +j 8.55558	12.0
1.5	0.075	3.19286 +j 8.46481	2.70264 +j 7.31695	13.8
2	0.1	2.50468 +j 6.21743	2.99129 +j 7.31405	15.1
2.5	0.125	2.79279 +j 6.2995	2.27501 +j 5.32718	15.9

Also, in order to calculate the efficiency of approximation (24) with respect to integral A (20) when calculating V'_{11} we have calculated error erA as follows:

$$erA = \left| \frac{A - A_{approx}}{A} \right| \cdot 100 . \qquad (32)$$

Some results togther with the calculated error erA are presented in table 2.

Although the calculated errors are higher than 5%, the total values of the approximate closed-form solutions of the Green's functions (27-28) are obtained more accurate. This shows that at small distances the direct term G_{11} is dominating. However, the results obtained by using new closed-form quasi-static approximative solutions are much more accurate than the results obtained by using modified images approximation (29-30) [6]. The calculated errors (in %) given in table 3 and 4 are defined by:

- $erG^{xx}_{AQSapprox}$ when comparing formulas (4) and (27) and erG^{xx}_{AIMG} when comparing formulas (5) and (28);

- $erG_{\phi QSapprox}$ when comparing formulas (4) and (29) and $erG_{\phi IMG}$ when comparing formulas (5) and (30).

Table 3 – *Error in % obtained by using formulas (27) and (29) forcalculating* G^{xx}_A

ρ (m)	ρ (w_{GR})	$erG^{xx}_{AQSapprox}$	erG^{xx}_{AIMG}
0.5	0.025	1.6	15.8
1	0.05	3.9	30.1
1.5	0.075	6.8	40.5
2	0.1	12.5	32.6
2.5	0.125	13.9	38.8

The comparison of the error obtained when calculating Green's function (5) by using quasi-static closed-form approximation (28) and image approximation (30) are given in table 4.

Table 4 – *Error in % obtained by using formulas (28) and (30) forcalculating* G_ϕ

ρ (m)	ρ (w_{GR})	$erG_{\phi QSapprox}$	$erG_{\phi IMG}$
0.5	0.025	3.9	10.9
1	0.05	7.2	19.1
1.5	0.075	9.6	26.5
2	0.1	10.2	47.3
2.5	0.125	11.4	51.1

The total value of the E_x component of the electric field is calculated by using numerical integration of (4-5) and by using approximate formulas (27-28) with error $erE_{xQSapprox}$ and (29-30) with error erE_{xIMG}. The results are presented in table 5.

Table 5 – *Error in % obtained when calculating* E_x

ρ (m)	ρ (w_{GR})	$erE_{xQSapprox}$	erE_{xIMG}
0.5	0.025	0.07	0.22
1	0.05	1.58	3.49
1.5	0.075	4.26	7.98
2	0.1	7.54	10.82
2.5	0.125	8.51	14.22

CONCLUSION

This paper presents a numerical study of possibilities for calculating Green's functions from horizontal Hertzian dipole in homogeneous ground by using quasi-static closed-form approximation. The preliminary results indicate that this approach is more accurate than the modified image method approach for radial distances < 0.2 of the wavelength in ground.

REFERENCES

[1] L.Grcev and F.Dawalibi, "An electromagnetic model for transients in grounding systems," *IEEE Transactions on Power Delivery*, Vol. PWRD-5, No. 4, October 1990, pp. 1773-1781.

[2] K. A. Michalski, "On the scalar potential of a point charge associated with time-harmonic dipole in a layered medium," *IEEE Trans. on Antennas and Propagation*, Vol. 35, Nov. 1987, pp. 1299-1301.

[3] A. Banos, "D*ipole radiation in the presence of a conducting half-space*", Pergamon Press, Oxford, 1966.

[4] P. Bannister, "Applications of complex image theory", Radio Science, Vol.21, No.4, pp.605-616, 1986.

[5] P. Bannister, , "Summary of Image Theory Expressions for the Quasi-Static Fields of Antennas at or Above Earth's Surface", Proc. IEEE, Vol.67, No.7, pp. 1001-1008, 1979.

[6] O. Gavriloska, M. Mircev, V. Arnautovski, L. Grcev, "Approximate image formulation for the fields of a horizontal Hertzian dipole in homogenous soil," *Proc. Int. PhD-Seminar "Numerical Field Computation and Optimization in Electrical Engineering*, Ohrid, Macedonia, 2005, pp. 89-94.