Regular paper

NUMERICAL SOLVING OF ELECTROSTATIC PROBLEMS OF TWO WIRE LINES

Milan M. DOBRIČIĆ[1], Jeroslav M. ŽIVANIĆ[2]

Abstract: In this paper we have presented a numerical method for determination of electrostatic values of two wire lines, which represents a mixture of charge simulation method.

Keywords: Conform mapping, Electric field intensity, charge density, Charge simulation method.

METHOD DESCRIPTION

In order to get the results as accurete as possible for the field itensity in the close area near the sharp edges of a conductor as well as on its edges, the author of this paper got an idea to use the charge simulation method and conform mappingof the isolated electrodes. That method, which has is the essential part of the combined method, has been presented i the author's master theisis. Firstly, mapping of the conductor's field of an outer part of a unit circle is done, in a way which was presented in the theisis. The contour which limits the cross-section of the conductor does the mapping in a unit circle. Then, we apply the method of charge simulation method in the same way as it is usually done, and we determine the values of fictive charge. Since the complex function of mapping has been determined, it is used for mapping of the points $\underline{z}_k$ where fictive charge can be found in the inner part of the second conductor into the points $\underline{w}_k$. Thus, we can get an equivalent electrostatic system in $\underline{w}-$plane which is made of a very long conductive cilinder with a circular unit circular cross-section and a stack of paralel conductors which are charged with loads per unit lengths put into the points $\underline{w}_k$. For the given system, according to the theory of reflection in a cilindral mirror, it is very easy to determine a complex potential and based on it, other values which are needed. This method gives very stable results for a field intensity and surface charge intensity on the surface of the conductor. That is firstly for the sharp edges where you can choose a point through which a contour is set, as a boundary of a cross-section f the conductor. The stability of the results is easily seen since the method does not change if the number of the used fictive sources changes. Thus, we have evaded the bad conditioning of the linear systems of the equations which always occurs with the method.

In the figure 1. we have presented a cross-section of a two wire line whose conductors are of a rectangular shape. The way of positioning of the fictive sources is also shown. Firstly, we make the sharp edges of a line rounded so that the radius of the edges is $r_0 > 0$. Secondly, the fictive sources are set in the inner space of the conductor which is drawn in the figure and presented by dashes.

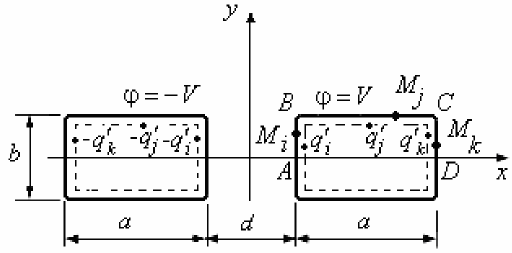

Fig.1. *Two wire line in* $\underline{Z}-$ *plane*.

The potential in the area of the wire line is

$$\varphi = \sum_{i_1=1}^{I} q'_{i_1} G(\boldsymbol{r}, \boldsymbol{r}'_i) + \sum_{j_1=1}^{J} q'_{j_1} G(\boldsymbol{r}, \boldsymbol{r}'_j) + \sum_{k_1=1}^{K} q'_{k_1} G(\boldsymbol{r}, \boldsymbol{r}'_k) \ , \qquad (1)$$

where $G(\boldsymbol{r}, \boldsymbol{r}'_i)$, $G(\boldsymbol{r}, \boldsymbol{r}'_j)$, $G(\boldsymbol{r}, \boldsymbol{r}'_k)$ are Green's functions.

After we give the boundary conditions for potential on the upper part of the right conductor in $I + J + K + 2$ because of the symmetry, we can calculate the point of the adjustment where we have taken into account points A and B, and thus we get a system of linear equations. By solving them we get the unknown fictive charge. Then, the other needed values can be calculated in a standard manner. However, the systems of equations are always badly conditioned which causes many problems with systems with more unknown values. The results tell us that the solution for the capacity per unit length is relatively stable since we are dealing with the integral values, while the solutions for the field intensity on sharp edges are not stable, i.e. they depend on the number of the used fictive sources. So, we do the following procedure:

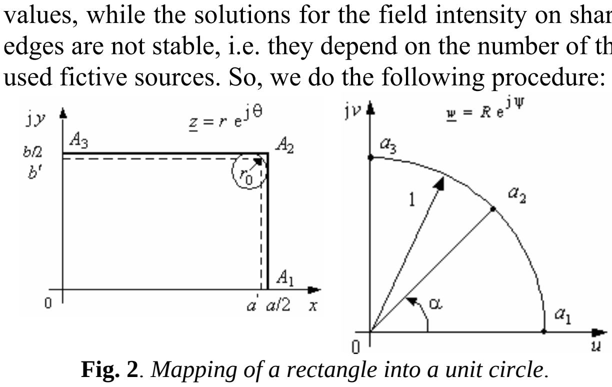

Fig. 2. *Mapping of a rectangle into a unit circle*.

[1] Applied Sciences Faculty of Cacak, Svetog Save 65, 32 000 Cacak, Serbia, E-mail:alenka@tfc.kg.ac.yu
[2] University of Kragujevac, Technical faculty Cacak, Svetog Save 65, 32000 Cacak, Serbia, E-mail: zivanic@ptt.yu

Firstly, we do a mapping of an outer left conductor, fig. 2 as in a detailed procedure shown [3]. Because of the symmetry of the figure, we will watch only one its fourth in the first quadrant, i.e. we form a functional

$$F=\int_0^{\alpha}\left(x-\frac{a}{2}\right)^2 \mathrm{d}\psi+\int_{\alpha}^{\pi/2}\left(y-\frac{b}{2}\right)^2 \mathrm{d}\psi+$$
$$+\lambda_x\left(\sum_{n=1}^{N} C_n\cos(m\alpha)-a'\right)+$$
$$+\lambda_y\left(\sum_{n=1}^{N} C_n\sin(m\alpha)-b'\right)=\min. \qquad (2)$$

where x and y represent parameters in the equations of the mapping functions [4]

$$x=\sum_{n=1}^{N} C_n\cos(m\psi)-\frac{a}{2}-\frac{d}{2}$$
$$y=\sum_{n=1}^{N} C_n\sin(m\psi) \qquad (3)$$

By minimizing of a functional (2) we get a system of non-linear equations. When solved, we get unknown values $C_n,\alpha,\lambda_x,\lambda_y$, where m represents the cardinal number of harmonica and for the rectangle it is $m=3-2n$. Thus, a complex function of mapping

$$\underline{z}=\sum_{n=1}^{N} C_n\,\underline{w}^m \qquad (4)$$

is determined by its parameters (3) where $\underline{z}=r\,\mathrm{e}^{\mathrm{j}\theta}$ and $\underline{w}=R\,\mathrm{e}^{\mathrm{j}\psi}$.

After we have determined the fictive charge as described above, an electrostatic system can be formed. It is made of a very long conductive cylinder of a unit circle radius as the picture of the left conductor in the $\underline{w}-$ plane, and the stack of fictive loads by which the right conductor has been replaced Fig.3.

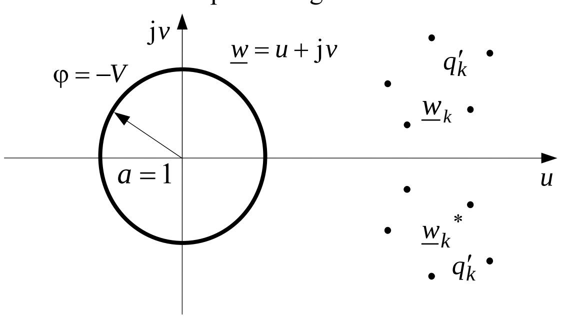

Fig. 3. *Equivalent electrostatic system in $\underline{w}-$plane*

Points $\underline{w}_k$ can be determined from the mapping function (4) since their position in the $\underline{z}-$plane known, and the function does the mapping presented with $\underline{z}=\underline{z}(\underline{w})$. Unfortunately , it is not possible to determine the inversion function $\underline{w}=\underline{w}(\underline{z})$ in its explicit form which means that for every fictive charge q'_k we should determine K of non-linear equations in order to specify their positions $\underline{w}_k$.

Since we want not to solve linear equations, we can apply a simple iterative procedure for points $\underline{w}_k$ determination starting from the mapping function (4) which develops into a Taylor's series in the nearness of every point $\underline{w}_k$ [4]

$$\underline{z}=\underline{z}_k+\underline{z}'_k\left(\underline{w}-\underline{w}_k\right) \qquad (5)$$

where

$$\underline{z}'_k=\left.\frac{\mathrm{d}\underline{z}}{\mathrm{d}\underline{w}}\right|_{\underline{w}=\underline{w}_k} \qquad (6)$$

that is

$$\underline{z}'_k=\sum_{n=1}^{N} m\,C_n\,\underline{w}_k^{\,m-1} \qquad (7)$$

and the derivation of a higher level has been neglected.

When the equation (7) is replaced with (5) and represented in $\underline{w}$, we get the formula

$$\underline{w}_k^{(i+1)}=\underline{w}_k^{(i)}+\frac{\sum_{n=1}^{N} C_n\,\underline{w}_k^{(i)m}-\underline{z}_k}{\sum_{n=1}^{N} m\,C_n\,\underline{w}_k^{(i)m-1}} \qquad (8)$$

where $i=0,1,..I$. The starting value $\underline{w}_k^{(0)}$ in the iterative process (8) is calculated for the value $|\underline{w}|\to\infty$ so, $\underline{z}=C_1\,\underline{w}$ and that means that

$$\underline{w}_k^{(0)}=\frac{\underline{z}_k}{C_1}. \qquad (9)$$

The suggested iterative procedure has a very swift convergence so that only after two or three iterative operations we get the values $\underline{w}_k$ with the eight decimal spaces accuracy.

Finally, when all the values have been determined applying the reflection theory in a cylinder mirror [5], a complex potential of a system on Fig. 3 is

$$\underline{\Phi}=-V+\frac{1}{2\pi\varepsilon_0}\sum_{k=1}^{K} q'_k\log\left(\frac{\underline{w}_k}{a}\right)^2\frac{\underline{w}-\underline{w}'_k}{\underline{w}-\underline{w}_k}\,\frac{\underline{w}-\underline{w}'^{*}_k}{\underline{w}-\underline{w}_k^{*}}. \qquad (10)$$

The positions of the fictive charge sources in the upper and in down half-plane are marked with $\underline{w}_k$ i.e. $\underline{w}_k^{*}$ and they are

$$\underline{w}_k=R_k\,\mathrm{e}^{\mathrm{j}\psi_k} \quad\text{and}\quad \underline{w}_k^{*}=R_k\,\mathrm{e}^{-\mathrm{j}\psi_k}\,, \qquad (11)$$

and their reflections in a mirror in the both upper and down half-planes are respectively

$$\underline{w}'_k=\frac{a^2}{R_k}\mathrm{e}^{\mathrm{j}\psi_k} \quad\text{and}\quad \underline{w}'^{*}_k=\frac{a^2}{R_k}\mathrm{e}^{-\mathrm{j}\psi_k} \qquad (12)$$

where $a = 1$ cross-section of a unit circle $\underline{w} = R\,e^{j\psi}$.

When a complex potential of the field intensity is determined in the nearness of a two wire line, in the $\underline{z}-$plane is calculated similarly [5]

$$E = \left|\frac{d\underline{\Phi}}{d\underline{w}}\right| \cdot \frac{1}{\left|\frac{d\underline{z}}{d\underline{w}}\right|} \quad (13)$$

where

$$\frac{d\underline{z}}{d\underline{w}} = \sum_{n=1}^{N} m C_n \underline{w}^{m-1}, \quad (14)$$

and his module

$$\left|\frac{d\underline{z}}{d\underline{w}}\right| = \sqrt{S_1{}^2 + S_2{}^2}\,. \quad (15)$$

values S_1 and S_2 are respectively

$$S_1 = \sum_{n=1}^{N} m C_n R^{m-1} \cos[(m-1)\psi]$$

and

$$S_2 = \sum_{n=1}^{N} m C_n R^{m-1} \sin[(m-1)\psi]. \quad (16)$$

Thus, for example, if we take that $R = 1$ and put it into the field intensity equation (13) and if the angle ψ is given in the limits of $0 \le \psi \le 2\pi$, we get the distribution of the field intensity on the surface of the conductor. Due to the present symmetry the angle ψ is to be limited $0 \le \psi \le \pi$. Numerical results for the field intensity calculated in this way are extremely stable especially on the sharp edges of the conductor. That means that the method is rather insensitive to the number of the used fictive sources. So, compared to the classical charge simulation method, this one has an advantage which is to be illustrated below.

Capacitance per unit length can be calculated using the charge simulation method since the values of the charged fictive sources are calculated, thus

$$C' = \frac{2\sum_{k=1}^{K} q'_k}{U}, \quad (17)$$

where $U-$ voltage between the conductors A.

Apart from this method, capacitance per length unit can be calculated by integrating of electric field on the surface of a conductive cylinder in the $\underline{w}-$plane. Since during the mapping the capacitance per length unit remains invariable value, and since the analytical equation for field intensity in $\underline{w}-$plane

$$E = \left|\frac{d\underline{\Phi}}{d\underline{w}}\right|, \quad (18)$$

normalized capacitance per length unit can be determined like

$$\frac{C'}{\varepsilon} = \int_0^{\pi} \left|\frac{d\underline{\Phi}}{d\underline{w}}\right| d\psi\,. \quad (17)$$

EXAMPLES

Two wire line of a rectangular cross-section Fig.1. For the dimensions of the line $a/b = 2$ and $r_0/a = 0{,}01$, after the above procedure is done, the unknown coefficients of a mapping function are (4)

Table I

The unknown	Unknown values
$\frac{C_n}{a}$	0,4369582714384
	0,1341658851947
	-0,0654552756117
	-0,0118735800185
	0,0035662819083
	0,0036509572965
	0,0002645563842
	-0,0012143163929
	-0,0006000300568
	0,0002881276592
α	0,6286630046151

Thus the mapping function is determined using the parameters (3), which constantly, point by point, does the mapping of the rectangle which limits the left conductor cross-section to a unit circle in the $\underline{w}-$plane, like in the Fig 4.

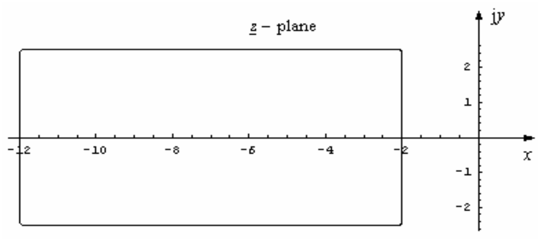

Fig.4- *Rectangle shape.*

Then, the position of the fictive charges in $\underline{w}-$plane $\underline{w}_i$, $\underline{w}_j$ and $\underline{w}_k$ can be determined, applying the above procedure (8). Thus we get a complex potential function (10) with the electric field intensity (13).

Table II

Value	$M = 10$		$M = 20$		$M = 50$	
	CSM	Comb. met.	CSM	Comb. met.	CSM	Comb. met.
$\frac{E_B}{E_A}$	3,7990	2,9477	3,0307	2,9487	2,7129	2,9494
$\frac{E_C}{E_A}$	1,2237	0,8462	0,9574	0,8353	0,8591	0,8312
$\frac{E_D}{E_A}$	0,2199	0,1914	0,2181	0,1888	0,2206	0,1879
$\frac{C'}{\varepsilon}$	3,0395	3,0395	3,0596	3,0596	3,0641	3,0641

Values of the field intensity in the points *A, B, C* and *D* for the line dimensions $a/b=2$, $d/a=0{,}5$ and $r_0/a=0{,}01$ are presented in the Table II, calculated according to the charge simulation method and a mixed method.

In the figure 5. A graph of the normalized electric field value on the surface of the conductor is given, based on Table II results

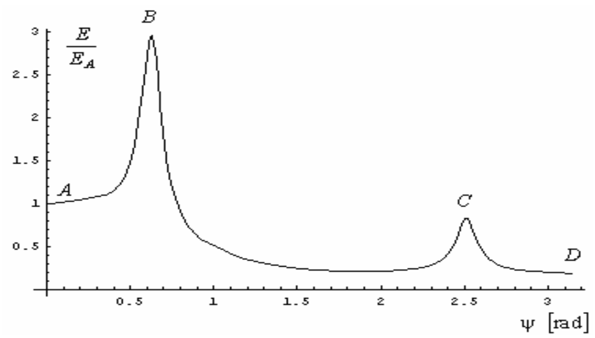

Fig. 5 – *Field intensity graph on the surface of a conductor*.

According to the results in the Table II. we can conclude that more stable solutions for he field intensity can be given by applying of a combined method, so it is much more suitable than CSM. Especially, if the distance between the conductors is lessen, since the speed comes into the first plan then, Table III for $a/b=2$, $d/a=0{,}05$ and $r_0/a=0{,}01$.

Table III

Normalized values of filed intensity and capacitance per unit length.

Value	$M=10$		$M=20$	
	CSM	Comb. method	CSM	Comb. method
$\frac{E_B}{E_A}$	1,7856	1,4733	1,3976	1,3977
$\frac{E_C}{E_A}$	0,1446	0,1011	0,1084	0,0960
$\frac{E_D}{E_A}$	0,0255	0,0224	0,0242	0,0213
C'/ε	13,166	13,166	13,373	13,373

Value	$M=50$		$M=100$	
	CSM	Comb. method	CSM	Comb. method
$\frac{E_B}{E_A}$	1,2943	1,3670	1,2219	1,3353
$\frac{E_C}{E_A}$	0,0991	0,0959	0,0953	0,0957
$\frac{E_D}{E_A}$	0,0234	0,0213	0,0220	0,0212
C'/ε	13,370	13,370	13,321	13,321

In the Fig 6 a graph of the normalized electric field value on the surface of the conductor is given for dimension given in Table III.

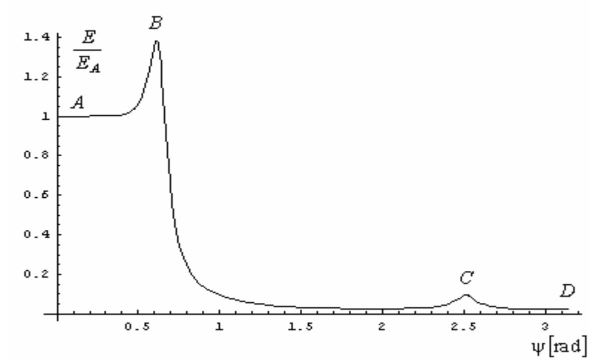

Fig.6- *Field intensity graph on the surface of the conductor*.

CONCLUSIONS

According to the results presented here, for the calculations of a capacitance per length unit of two wire lines, we can apply CSM with great accuracy, whilst for the calculations for the field intensity a mixed method is recommended especially if the small distance between the conductors and sharp edges are given.

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Mr Milan Dobričić was born on 3rd September, 1962 in Čačak. He graduated from Technical Faculty in Čačak, Department of Electrical engineering in 1987. The same year he got a job in a secondary technical school in Cacak and in 1989 he started working at Technical College where he has been employed as an associate worker for the subjects: Electrical machines and Electro motor drives with regulation.

He defended his master thesis – "About approximate numerical solving of electrostatic problems with the help of conformal transformation" - at Electronic Faculty in Nis.

His field of interest is Theory of fields. He is working on his doctoral thesis under the guidance of Prof Dr J. Zivanic from the field of application of function of complex variable.